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OHM'S LAW IN THE FAST LANE: GENERAL RELATIVISTIC CHARGE DYNAMICS

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ABSTRACT

Fully relativistic and causal equations for the flow of charge in curved spacetime are derived. It is believed that this is the first set of equations to be published that correctly describes the flow of charge, as well as the evolution of the electromagnetic field, in highly dynamical relativistic environments on timescales much shorter than the collapse time (GM/c^3). The equations will therefore be important for correctly investigating problems such as the dynamical collapse of magnetized stellar cores to black holes and the production of jets. Both are potentially important problems in the study of gamma-ray burst engine models and in predicting the dynamical morphology of the collapse and the character of the gravitational waves generated. This system of equations, given the name of "charge dynamics," is analogous to those of hydrodynamics (which describe the flow of *mass* in spacetime rather than the flow of charge). The most important equation in the system is the relativistic generalized Ohm's law, which is used to compute time-dependent four-current. Unlike other equations for the current now in use, this one ensures that charge drift velocities remain less than the speed of light, takes into account the finite current rise time, is expressed in a covariant form, and is suitable for general relativistic computations in an arbitrary metric. It includes the standard known effects (Lorentz force, Hall effect, pressure effect, and resistivity) and reduces to known forms of Ohm's law in the appropriate limits. In addition, the plasma particles are allowed to have highly relativistic drift velocities, resulting in an implicit equation for the "current beaming factor" γ_q. It is proposed that, short of solving the multifluid plasma equations or the relativistic Boltzmann equation itself, these are the most general expressions for relativistic current flow in the one-fluid approximation, and they should be made part of the general set of equations that are solved in extreme black hole accretion and fully general numerical relativistic collapse simulations.

Subject headings: black hole physics — gamma rays: bursts — MHD — relativity

1. INTRODUCTION

This paper is another in a series whose goal is to establish the mathematical, physical, and numerical tools necessary to understand and simulate the formation of black holes and the production (through electrodynamic processes) of relativistic jets during that collapse. The full understanding of how black holes are formed, and how jets and gravitational waves might be generated in accretion and gravitational collapse, is currently one of the most challenging, and far-reaching, astrophysical problems. It has important observational consequences for gravitational wave sources, gamma-ray bursts (GRBs), quasars, and microquasars. Its solution will involve nearly every branch of theoretical astrophysics (nuclear and particle physics, electromagnetics, gravity [numerical relativity], plasma flow, radiation transport, and dynamics). In addition, the formulation of the problem will require that these processes be expressed in a general relativistic framework that respects the principles of causality and covariance on timescales considerably shorter than the light crossing time of the forming black hole ($\ll GM/c^3$). While there is a good understanding of how charge behaves near black holes in equilibrium situations, i.e., on times $\gg GM/c^3$ (Wald 1974; Lee, Lee, & van Putten 2001), to date there appears to be no thorough treatment of charge flow and field evolution in strong gravity on very *short* timescales. Such a treatment should take into account the fact that the current rise time can be long compared to the characteristic timescale, properly account for particle velocities (bulk, drift, and thermal) that can approach c, be expressible in an arbitrary spacetime metric, and also reproduce, in the appropriate limits, the standard known effects (including the Hall and pressure effects, not just electric acceleration, Faraday induction, and ohmic resistivity).

The purpose of this paper is to return to the basic equations of general relativistic statistical mechanics and properly derive the covariant equations for the evolution of the current. Section 2 discusses the framework of the problem and the present lack of a good equation for the current in highly relativistic situations. Section 3 sets up the relativistic Boltzmann problem, and § 4 derives the relativistic plasma equations. Section 5 discusses relativistic one-fluid plasma theory and derives the generalized Ohm's law and charge dynamics. Section 6 shows that the theory reduces to various previous generalized Ohm's laws in the appropriate limits and discusses the applicability of the one-fluid theory. The most important results of this paper are the basic equations of charge dynamics (56)–(58) and (63) or, in component form, equations (68)–(71).

2. CAUSAL AND COVARIANCE PROBLEMS IN PRESENT TREATMENTS OF THE CURRENT

The two governing sets of equations for the general numerical relativity problem of electromagnetic black hole formation are the Einstein equations

$$\mathbf{G} = \frac{8\pi G}{c^4}\mathbf{T} \tag{1}$$

for the gravitational field and the Maxwell equations

$$\nabla \cdot \mathbf{F} = \frac{4\pi}{c}\mathbf{J}, \tag{2}$$

$$\nabla \cdot \mathbf{M} = 0 \tag{3}$$

(Faraday's and Ampere's laws) for the electromagnetic field, where $\nabla = \mathbf{e}_\mu = \partial/\partial x^\mu$ is the four-gradient operator; $\mathbf{G}$ is the second-rank, symmetric Einstein tensor that describes the second derivatives (curvature) of the metric $\mathbf{g}$; and $\mathbf{F}$ is the second-rank, *anti*symmetric Faraday tensor that describes the electromagnetic field. Here $\mathbf{T}$ is the energy-momentum-stress tensor that serves as the source for gravity and $\mathbf{J}$ is the four-current vector that serves as the source of the electromagnetic field. The Maxwell tensor $\mathbf{M} = {}^*\mathbf{F}$ (the dual of the Faraday tensor), has the same information as the Faraday tensor and also is antisymmetric, but has the roles of electric and magnetic field reversed. Because of their geometric properties, these tensors satisfy the following Bianchi identities

$$\nabla \cdot \mathbf{G} = 0, \tag{4}$$

$$\nabla \cdot (\nabla \cdot \mathbf{F}) = 0, \tag{5}$$

$$\nabla \cdot (\nabla \cdot \mathbf{M}) = 0, \tag{6}$$

which, from equations (1)–(3), give rise to the conservation laws of energy and momentum,

$$\nabla \cdot \mathbf{T} = 0, \tag{7}$$

and of charge,

$$\nabla \cdot \mathbf{J} = 0. \tag{8}$$

The solution of equation (7) gives the distribution of temperature T and four-velocity $\mathbf{U}(\mathbf{x})$, which is constrained to always have the absolute value of the speed of light,

$$\mathbf{U} \cdot \mathbf{U} = -c^2.$$

The addition of particle/rest-mass conservation,

$$\nabla \cdot \rho_m \mathbf{U} = 0, \tag{9}$$

where ρ_m is the rest mass density in the fluid frame, allows a solution for $\rho_m(\mathbf{x})$ to be found as well. Together, $T(\mathbf{x})$ and $\rho_m(\mathbf{x})$ can be used to compute the state variables (pressure, internal energy, etc.) that close the energy and momentum conservation laws.

The solution to equation (8), however, which can be written as

$$\nabla \cdot (\rho_q \mathbf{U} + \mathbf{j}) = 0,$$

gives only one quantity, the charge density ρ_q in the fluid frame, in terms of the spatial charge current

$$\mathbf{j} = \mathbf{P} \cdot \mathbf{J}, \tag{10}$$

where

$$\mathbf{P} \equiv \frac{1}{c^2}\mathbf{U} \otimes \mathbf{U} + \mathbf{g} \tag{11}$$

is the spatial projection tensor orthogonal to the four-velocity unit vector $\mathbf{e}_\mathbf{U} \equiv \mathbf{U}/c$. The symbol $\otimes$ signifies the outer tensor product. While $\mathbf{j}$ is a four-vector, it is constrained to have only three independent components by its orthogonality to $\mathbf{U}$,

$$\mathbf{U} \cdot \mathbf{j} = 0. \tag{12}$$

However, *none* of the above equations can be used to specify these three components of the current.

Previous treatments of black hole electrodynamics have made a variety of assumptions to determine $\mathbf{j}$ and thereby close the electromagnetic equations. One popular technique is to assume that the electromagnetic field dominates the dynamics and is time-independent, leading to the "force-free" condition

$$\mathbf{J} \cdot \mathbf{F} = 0. \tag{13}$$

This assumption has been criticized (Punsly 2003) as leading to effects that violate causality. That is, the force-free condition is *acausal*, and therefore not relativistically acceptable for forming black holes and GRBs.

Another approach has been to assume general relativistic magnetohydrodynamics, as outlined by many in the past (Lichnerowicz 1967; Eckart 1940; Anandan 1984; Blackman & Field 1993). This approach relates the current to the electromagnetic field through a simple form of Ohm's law,

$$\eta \mathbf{j} = \frac{\mathbf{U}}{c} \cdot \mathbf{F}. \tag{14}$$

where $\eta(\mathbf{x})$ is the resistivity distribution of the plasma. In the nonrelativistic limit, this takes on the familiar form

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \times \boldsymbol{B}\right),$$

where $\sigma = 1/\eta$ is the plasma conductivity and $\boldsymbol{J}$, $\boldsymbol{E}$, $\boldsymbol{V}$, and $\boldsymbol{B}$ are the current, electric field, velocity, and magnetic field three-vectors.

Ideal relativistic MHD is a further simplification current flow that is useful for highly conducting plasmas, such as those in most astrophysical situations. With $\eta \to 0$, Ohm's law reduces to

$$\mathbf{U} \cdot \mathbf{F} = 0, \tag{15}$$

which, in the nonrelativistic limit, becomes

$$\boldsymbol{E} = -\frac{\boldsymbol{V}}{c} \times \boldsymbol{B}.$$

The ideal MHD condition is not so much an equation for the current as a condition on components of the Faraday tensor (the electric field): when the conductivity is high, the local electric field in the plasma shorts out, leaving only the EMF due to charged plasma motion in the magnetic field. The current itself is determined by first determining the electromagnetic field from equation (3) and then inverting equation (2) for $\mathbf{J}$. Equations (13) and (15) appear very similar. They both state that the Faraday tensor is orthogonal to a four-velocity, either the particle drift velocity or the average particle velocity. However, they generate very different physics.

Unfortunately, a criticism that is equally harsh as that of the force-free condition can be leveled against the MHD condition, even the resistive version of it (eq. [14]). The latter states that the application of an electromagnetic field instantaneously generates a current. However, this also is acausal. There is no immediacy in relativistic dynamics. A current builds up after a finite, albeit short, rise time. And in relativistic flow, the time-dependent increase in a current might be interrupted by any number of other rapid phenomena, resulting in possible charge separation and time-dependent charge dynamics.

A nonrelativistic version of the "generalized Ohm's law" is often used in laboratory plasma physics and is given by the expression (Rossi & Olbert 1970; Krall & Trivelpiece 1973)

$$\frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot (\boldsymbol{VJ} + \boldsymbol{JV} - \rho_q \boldsymbol{VV}) + \nabla p_q = \ell\left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \times \boldsymbol{B} + h\frac{\boldsymbol{J}}{c} \times \mathbf{B} - \eta \boldsymbol{J}\right), \tag{16}$$

where ∇ is the three-space gradient operator and the charge-weighted pressure (per unit mass) is

$$p_q = \sum_a q_a p_a / m_a$$

q_a is that particle's charge, m_a is its mass, and p_a is that species' partial pressure. The Lorentz, Hall, and resistivity coefficients are

$$\ell = \sum_a n_a q_a^2 / m_a,$$
$$h = \frac{1}{\ell}\sum_a q_a / m_a,$$
$$\eta = \nu / \ell,$$

where n_a is the number density of particle species a. The resistivity term results from integrating particle collisions over velocity and approximating the result as an effective collision frequency ν,

$$\sum_a n_a q_a \int_{V_v} \boldsymbol{v} \dot{f}_{a,\mathrm{coll}}\, d^3\boldsymbol{v} \equiv -\nu \boldsymbol{J}.$$

Equation (16) has most of the effects we are looking for in a description of charge dynamics (finite current rise time, Lorentz force, Hall effect, pressure effect, and resistivity). However, it is valid only to linear order in $\boldsymbol{V}/c$ (still acausal), only in the laboratory coordinate system (not covariant), and only in flat space.

Ardavan (1976) derived a relativistic form of Ohm's law for a cold plasma (vanishing p_a and p_q) in flat space. This expression turns out to have some errors, but is still useful for checking our results in § 5. (Expressions for a relativistic cold *pair* plasma also have been derived [Gedalin 1996; Melatos & Melrose 1996], but the Ardavan expression is a little more general and therefore more useful to us.) In our notation the corrected Ardavan equation for the current is

$$\begin{aligned}
&\frac{\partial}{c\partial t}[\mathrm{U}^0\, {}^3\mathbf{J} + (\mathrm{J}^0 - \rho_q \mathrm{U}^0)^3\mathbf{U}] \\
&+ \nabla \cdot [{}^3\mathbf{J}\, {}^3\mathbf{U} + {}^3\mathbf{U}\, {}^3\mathbf{J} - \rho_q\, {}^3\mathbf{U}\, {}^3\mathbf{U}] \\
&= \ell\left\{\frac{1}{c}[\mathrm{U}^0\boldsymbol{E} + {}^3\mathbf{U} \times \boldsymbol{B}] \right.\\
&\quad \left. + h\frac{1}{c}\left[(\mathrm{J}^0 - \rho_q \mathrm{U}^0)\boldsymbol{E} + {}^3\mathbf{J} \times \boldsymbol{B}\right] - \eta^3\mathbf{J}\right\},
\end{aligned} \tag{17}$$

where U^0 and J^0 are the temporal components of the four-velocity and four-current, respectively; ${}^3\mathbf{U}$ and ${}^3\mathbf{J}$ are the spatial three-vector components of those four-vectors (i.e., ${}^3\mathbf{U} = \gamma \boldsymbol{V} = \mathrm{U}^0 \boldsymbol{V}/c$). The errors that have been corrected, all on the right-hand side, are a sign error in the Lorentz term and the addition of $-h\rho_q \mathrm{U}^0 \boldsymbol{E}/c$ in the Hall term. In the limit of nonrelativistic flow, $\mathrm{U}^0 \to c$, $\mathrm{J}^0 \to \rho_q c$, ${}^3\mathbf{U} \to \boldsymbol{V}$, and ${}^3\mathbf{J} \to \boldsymbol{J}$. So, with these corrections, equation (17) reduces to equation (16) when the charge-weighted pressure $p_q = 0$. Equation (17) is both causal and covariant, but it is valid only for Lorentz systems in flat space and only when the plasma is truly cold. It does not include effects that occur when the plasma has a relativistic temperature or a relativistic current drift velocity.

The goal of this paper is to derive a description of charge flow that is valid in all relativistic situations—relativistic bulk flow, hot plasma, relativistic current drift velocities—and that is causal, covariant, and valid in any spacetime metric.

3. GENERAL RELATIVISTIC STATISTICAL MECHANICS

3.1. *Phase Space and Particle Density*

Phase space $\Omega = \Omega_x \otimes \Omega_u$ is inherently eight-dimensional, not six. The generalized coordinates are the position and four-velocity $(\mathrm{x}^\mu,\ \mathrm{u}^\nu)$, where $\mu,\ \nu = 0, 1, 2, 3$. The Ω_x is a general curved spacetime with a global time coordinate $x^0 = ct$ and three spatial coordinates. The volume element

$$d\Omega_x = \sqrt{-g}\, d\mathrm{x}^0\, d\mathrm{x}^1\, d\mathrm{x}^2\, d\mathrm{x}^3$$

(where $\sqrt{-g}$ is the determinant of the spacetime metric) is an invariant over the entire spacetime. Therefore, in a general metric, $d\Omega_x$ cannot be separated into globally invariant temporal and spatial parts. However, in each local Lorentz frame at a given point in spacetime, it *can* be separated as

$$d\hat{\Omega}_x = d\tau\, d\Upsilon_x,$$

where

$$d\tau \equiv d\mathrm{x}^{\hat{0}}/(\mathrm{u}^{\hat{0}}/c),$$
$$d\Upsilon_x \equiv (\mathrm{u}^{\hat{0}}/c) d\mathrm{x}^{\hat{1}} d\mathrm{x}^{\hat{2}} d\mathrm{x}^{\hat{3}}.$$

Each of the factors in equation (18) are Lorentz invariant (Misner, Thorne, & Wheeler 1973). We therefore can define a Lorentz invariant *three-space* density of particles of species a and four-velocity $\mathbf{u}$ in each local Lorentz frame

$$\aleph_a \equiv \frac{d\mathrm{N}_a}{d\Upsilon_x d\hat{\Omega}_u} \tag{18}$$

in the *seven*-dimensional phase space $\Upsilon_x \otimes \hat{\Omega}_u$.

While $\hat{\Omega}_u$ is a four-dimensional velocity space, all particles are constrained to move on a three-dimensional hypersurface within that space (the "mass hyperboloid") Υ_u, defined by the normalization of particle velocity

$$\mathbf{u} \cdot \mathbf{u} = -c^2. \tag{19}$$

In regions of $\hat{\Omega}_u$ outside of Υ_u the particle density vanishes, as there are no particles with $\mathbf{u} \cdot \mathbf{u} \neq -c^2$. Some treatments of relativistic statistical mechanics incorporate the constraint (19) into the Boltzmann equation directly. Indeed, one can separate

$d\hat{\Omega}_u$ into temporal and spatial parts, in a manner similar to $d\hat{\Omega}_x$,

$$d\hat{\Omega}_u = d\epsilon\, d\Upsilon_u, \tag{20}$$

where

$$d\epsilon \equiv (\mathrm{u}^{\hat{0}}/c)d\mathrm{u} = d(\mathrm{u}^{\hat{0}})^2/2c, \tag{21}$$

$$d\Upsilon_u \equiv d\mathrm{u}^{\hat{1}}d\mathrm{u}^{\hat{2}}d\mathrm{u}^{\hat{3}}/(\mathrm{u}^{\hat{0}}/c). \tag{22}$$

Not only are $d\epsilon$ and $d\Upsilon_u$ Lorentz invariant, the product

$$d\Upsilon_x d\Upsilon_u = d\mathrm{x}^{\hat{1}}d\mathrm{x}^{\hat{2}}d\mathrm{x}^{\hat{3}}d\mathrm{u}^{\hat{1}}d\mathrm{u}^{\hat{2}}d\mathrm{u}^{\hat{3}}$$

is also Lorentz invariant (Misner et al. 1973). One therefore *could* define an invariant density in *six*-dimensional phase space instead of $\aleph_a$. However, in this paper it has been found to be more useful to use the seven-dimensional phase space and perform the velocity integrals over $\hat{\Omega}_u$. The mass hyperboloid then is enforced only at the end of the computation when the integrals are evaluated. This is accomplished by using a delta function to describe the lack of particles outside of the mass shell in the distribution $\aleph_a$. (See Appendix.)

A final property of the particle density to note is that, as $|\mathrm{u}^i| \to \infty$ (where $i = 1, 2, 3$), $\aleph_a$ approaches 0 faster than any power of u^i. Therefore, $\aleph_a$, multiplied by any power of u^i, vanishes on the hypersurface $\partial\hat{\Omega}_u$ (the boundary of $\hat{\Omega}_u$).

3.2. *The Relativistic Boltzmann Equation*

The density $\aleph_a$ for each particle species obeys the relativistic Boltzmann equation

$$\frac{d\aleph_a}{d\tau} = \dot{\aleph}_{a,\mathrm{col}}, \tag{23}$$

where τ again is the proper time for particles in that region of phase space, and $\dot{\aleph}_{a,\mathrm{col}}$ is the number density of collisions per unit time of that particle species at that point in phase space (summed over all other particles of all other species)

$$\dot{\aleph}_{a,\mathrm{col}} = -\sum_b \int_{\Omega'} (\mathbf{a} - \mathbf{a}') \cdot \nabla_u \aleph'_{ab}(\mathbf{x}, \mathbf{u}, \mathbf{x}', \mathbf{u}')d\Omega', \tag{24}$$

where $\mathbf{a}$ is the particle acceleration caused by body forces (i.e., forces other than particle collisions). Equation (23) can be rewritten in generalized coordinates as

$$\dot{s}^r \frac{\partial\aleph_a}{\partial s^r} = \dot{\aleph}_{a,\mathrm{col}},$$

where $s^r \equiv (\mathrm{x}^\mu,\ \mathrm{u}^\nu)$ with $r = 0, 1, \ldots, 6, 7$, or, in geometric form,

$$\mathbf{u} \cdot \nabla_x \aleph_a + \mathbf{a} \cdot \nabla_u \aleph_a = \dot{\aleph}_{a,\mathrm{col}}, \tag{25}$$

and the gradient operators are

$$\nabla_x \equiv \mathbf{e}_x \equiv \frac{\partial}{\partial \mathbf{x}},$$

$$\nabla_u \equiv \mathbf{e}_u \equiv \frac{\partial}{\partial \mathbf{u}}.$$

For the electromagnetic field, $\mathbf{a}$ is the Lorentz acceleration

$$\mathbf{a} = \frac{q_a}{m_a c}\mathbf{u} \cdot \mathbf{F}.$$

Some treatments of general relativistic statistical mechanics (Andréasson 2002)[1] also include the gravitational "force" by including Christoffel symbols in the acceleration. However, this is not necessary, nor really desirable, as we can implicitly take these effects into account by using only the geometrical form of the gradient operator throughout the derivation and then using the equivalence principal at the end to convert the equations to component form. As a body force, gravity is automatically included in the structure of any spacetime in which ∇_x is evaluated.

4. THE MULTIFLUID EQUATIONS

The equations of plasma dynamics are generated by taking velocity moments of the relativistic Boltzmann equation, which gives rise to many hydrodynamic and thermodynamic quantities. Taking the moment involves multiplying equation (25) by a power of the velocity *coordinate* vector $\mathbf{u}$ and integrating over all velocity space $\hat{\Omega}_u$.

4.1. *The Zeroth Moment: Conservation of Particles*

Multiplying equation (25) by unity and integrating over $\hat{\Omega}_u$ produces the zeroth moment. One can show that the integral of the second term in that equation vanishes by first integrating by parts,

$$\int_{\hat{\Omega}_u} \mathbf{a} \cdot \nabla_u \aleph_a\, d\hat{\Omega}_u = \int_{\hat{\Omega}_u} \nabla_u \cdot (\mathbf{a}\aleph_a)\, d\hat{\Omega}_u - \int_{\hat{\Omega}_u} (\nabla_u \cdot \mathbf{a})\aleph_a\, d\hat{\Omega}_u. \tag{26}$$

Gauss' law then can be used to show that the first term in equation (26) vanishes because $\aleph_a$ vanishes on the boundary $\partial\hat{\Omega}_u$,

$$\int_{\hat{\Omega}_u} \nabla_u \cdot (\mathbf{a}\aleph_a) d\hat{\Omega}_u = \oint_{\partial\hat{\Omega}_u} \aleph_a \mathbf{a} \cdot d\mathbf{\Sigma}_u = 0,$$

where $d\mathbf{\Sigma}_u$ is the three-volume element on $\partial\hat{\Omega}_u$. The kernel $\nabla_u \cdot \mathbf{a}$ in the second term in equation (26) also vanishes because the Faraday tensor is antisymmetric and independent of the velocity coordinate,[2]

$$\nabla_u \cdot \left(\frac{q_a \mathbf{u} \cdot \mathbf{F}}{m_a c}\right) = 0.$$

Similarly, the velocity integral of the right-hand side of equation (25) vanishes because $\dot{\aleph}_{a,\mathrm{col}}$ is in a similar form to the $\mathbf{a} \cdot \nabla_u \aleph_a$ term in equation (25) (see eq. [24]). Because $\mathbf{x}$ and $\mathbf{u}$ are independent generalized coordinates, $\nabla_x \cdot \mathbf{u} = 0$, so the velocity integral of the *zeroth* moment of equation (25) becomes simply

$$\nabla_x \cdot \int_{\hat{\Omega}_u} \mathbf{u}\aleph_a\, d\hat{\Omega}_u = 0. \tag{27}$$

[1] Andréasson (2002) available at http://www.livingreviews.org/lrr-2002-7.
[2] In component notation, $(\partial/\partial\mathrm{u}^\mu)(\mathrm{u}^\nu F^\mu_\nu) = \delta^\nu_\mu F^\mu_\nu = F^\mu_\mu = 0$.

In order to interpret this equation, we need a coordinate gauge in which to express $\mathbf{u}$.

4.2. *Velocity Decomposition and the Velocity Coordinate Gauge*

We choose to decompose the velocity coordinate $\mathbf{u}$ into two components, one along the center–of–rest-mass average particle velocity $\mathbf{U}$,

$$\mathbf{U} \equiv \frac{\sum_a m_a \int_{\hat{\Omega}_u} \mathbf{u} \aleph_a \, d\hat{\Omega}_u}{\sum_a m_a \int_{\hat{\Omega}_u} \aleph_a \, d\hat{\Omega}_u} \tag{28}$$

(the justification for this choice of average velocity will be given later), and one orthogonal to $\mathbf{U}$ (the drift four-velocity $\mathbf{v}$), yielding

$$\mathbf{u} = \gamma(\mathbf{U} + \mathbf{v}), \tag{29}$$

where the particle Lorentz factor is

$$\gamma \equiv -\frac{1}{c^2}(\mathbf{U} \cdot \mathbf{u}) \tag{30}$$

and the relative spatial velocity coordinate is

$$\mathbf{v} \equiv (\mathbf{P} \cdot \mathbf{u})/\gamma. \tag{31}$$

Note that $\mathbf{v}$ is still a four-vector, but it is constrained to have only three independent components by its orthogonality with $\mathbf{U}$,

$$\mathbf{U} \cdot \mathbf{v} = 0. \tag{32}$$

It therefore is a velocity *coordinate* that spans Υ_u. In the rest frame of the fluid, the velocity components are

$$\mathbf{u} = \left(\gamma c, \ \gamma \mathrm{v}^1, \ \gamma \mathrm{v}^2, \ \gamma \mathrm{v}^3\right), \tag{33}$$

$$\mathbf{v} = \left(0, \ \mathrm{v}^1, \ \mathrm{v}^2, \ \mathrm{v}^3\right), \tag{34}$$

and the constraint (eq. [19]) on the particle velocity $\mathbf{u}$ becomes

$$\gamma = \left(1 - \mathbf{v} \cdot \mathbf{v}/c^2\right)^{-1/2},$$

as expected.

With the velocity decomposition in equation (29), the equation of continuity (27) for particle species a becomes

$$\nabla_x \cdot n_a(\mathbf{U} + \mathbf{V}_a) = 0, \tag{35}$$

where

$$n_a \equiv \int_{\hat{\Omega}_u} \gamma \aleph_a \, d\hat{\Omega}_u \tag{36}$$

is the particle density and

$$\mathbf{V}_a = \frac{1}{n_a} \int_{\hat{\Omega}_u} \gamma \mathbf{v} \aleph_a \, d\hat{\Omega}_u \tag{37}$$

is the average particle *drift* velocity for species a. The equation of continuity (35) is important for the conservation laws of rest mass and charge in one-fluid dynamics. Despite the Lorentz factor in the above velocity integrals, these quantities are, in fact, the familiar three-momentum integrals of standard thermodynamics, in which the single factor of γ does not appear (see Appendix).

4.3. *The First Moment: Conservation of Particle Energy-Momentum*

The first moment of equation (25) generates a vector equation

$$\nabla_x \cdot (\aleph_a \mathbf{u} \otimes \mathbf{u}) + \mathbf{u}\left\{\frac{q_a \mathbf{u} \cdot \mathbf{F}}{m_a c} \cdot \nabla_u \aleph_a\right\} = \mathbf{u} \dot{\aleph}_{a,\mathrm{col}}, \tag{38}$$

which also can be integrated over $d\hat{\Omega}_u$ to yield

$$\begin{aligned} \nabla_x \cdot \left[n'_a \mathbf{U} \otimes \mathbf{U} + n_a \mathbf{U} \otimes \mathbf{V}'_a + n_a \mathbf{V}'_a \otimes \mathbf{U} + \Pi_a \right] \\ = \frac{1}{m_a c} \mathbf{J}_a \cdot \mathbf{F} - \nu n_a (\mathbf{U} + \mathbf{V}_a), \end{aligned} \tag{39}$$

where

$$n'_a \equiv \int_{\hat{\Omega}_u} \gamma^2 \aleph_a \, d\hat{\Omega}_u, \tag{40}$$

$$\mathbf{V}'_a \equiv \frac{1}{n_a} \int_{\hat{\Omega}_u} \gamma^2 \mathbf{v} \aleph_a \, d\hat{\Omega}_u. \tag{41}$$

In deriving equation (39) we have substituted equation (29) for $\mathbf{u}$ in the first term of equation (38) and discarded a vanishing boundary integral that results from integrating the second term by parts.

Note the extra factor of γ in the integrals in equations (40) and (41) compared with (36) and (37). These are beamed quantities that give rise to relativistic internal energy, pressure, etc. The partial current is

$$\mathbf{J}_a \equiv q_a \int_{\hat{\Omega}_u} \mathbf{u} \aleph_a \, d\Omega = q_a n_a (\mathbf{U} + \mathbf{V}_a), \tag{42}$$

which has components $q_a n_a(c, \ \mathrm{V}_a^1, \ \mathrm{V}_a^2, \ \mathrm{V}_a^3)$ in the fluid rest frame. A four-vector

$$\mathbf{j}_a = q_a n_a \mathbf{V}_a \tag{43}$$

can be used to describe the $\mathbf{U}$-orthogonal part of the current (cf. eq. [10]), with $\mathbf{U} \cdot \mathbf{j}_a = 0$. The partial pressure tensor (per unit mass) of species a is

$$\begin{aligned} \Pi_a &\equiv \int_{\hat{\Omega}_u} (\mathbf{P} \cdot \mathbf{v}) \otimes (\mathbf{P} \cdot \mathbf{v}) \aleph_a \, d\hat{\Omega}_u \\ &= \int_{\hat{\Omega}_u} \gamma^2 (\mathbf{v} \otimes \mathbf{v}) \aleph_a \, d\hat{\Omega}_u. \end{aligned} \tag{44}$$

Note also in equation (39) that the collision term has been simplified to a collision frequency ν times the integrated particle flux.

5. THE ONE-FLUID EQUATIONS

At present, astrophysical simulation codes deal almost exclusively with the one-fluid equations when dynamics are concerned. That is, the individual particle species dynamical equations are summed and solved as a single set of equations. Of course, many stellar evolution and collapse codes track the composition for different species of particle, but this is usually done for equation of state and composition purposes only, not to determine relative drift velocities of different charged species, for example. Therefore, for at least the near future, it will still be important to construct one-fluid equations for studies of processes like black hole formation and accretion.

5.1. *The Example of Hydrodynamics*

Before deriving the equations of relativistic charge dynamics, it is important to review the derivation of the equations of hydrodynamics in the presence of an electromagnetic field. This is more than just an exercise in "reinventing the wheel." Reviewing this derivation will allow us to check that our equations and procedures are correct and define some quantities that will be needed later in the discussion of charge dynamics, and it will assist us in understanding the new set of equations in terms of the familiar hydrodynamic ones. The procedure is to simply weight the five equations (35)–(39) by the particle *rest* mass m_a and sum over all species,

$$\nabla_x \cdot \rho_m \mathbf{U} = 0, \tag{45}$$

$$\nabla_x \cdot \left\{ \left[\rho_m + \frac{\varepsilon}{c^2}\right] \mathbf{U} \otimes \mathbf{U} + \frac{1}{c^2}[\mathbf{U} \otimes \mathbf{H} + \mathbf{H} \otimes \mathbf{U}] + p\mathbf{P} \right\} = \frac{1}{c} \mathbf{J} \cdot \mathbf{F}, \tag{46}$$

where

$$\rho_m \equiv \sum_a n_a m_a \tag{47}$$

is the total *rest* mass density,

$$\varepsilon \equiv \sum_a \varepsilon_a \equiv \sum_a (n_a' - n_a) m_a c^2 = \sum_a m_a c^2 \int_{\hat{\Omega}_u} \gamma(\gamma - 1) \aleph_a \, d\hat{\Omega}_u \tag{48}$$

is the internal (kinetic) energy density,

$$\mathbf{H} \equiv \sum_a n_a m_a c^2 \mathbf{V}_a' \tag{49}$$

is the heat flux (including relativistic enhancement), and

$$p \equiv \sum_a p_a \equiv \sum_a \frac{m_a}{3} \int_{\hat{\Omega}_u} \gamma^2 (\mathbf{v} \cdot \mathbf{v}) \aleph_a \, d\hat{\Omega}_u \tag{50}$$

is the scalar pressure for an *isotropic* distribution in Υ_u. Equation (45) is correct only if the rest-mass–centered drift velocity is zero,

$$\sum_a n_a m_a \mathbf{V}_a = 0,$$

and that is the case only if the average velocity is defined according to equation (28), thereby justifying our rest-mass–centered choice for $\mathbf{U}$.

We can show that equation (46) is equivalent to equation (7) if we define the following energy-momentum-stress tensor:[3]

$$\mathbf{T} \equiv \mathbf{T}^{FL} + \mathbf{T},^{EM} \tag{51}$$

$$\mathbf{T}^{FL} \equiv \left(\rho_m + \frac{\varepsilon + p}{c^2}\right) \mathbf{U} \otimes \mathbf{U} + \frac{1}{c^2}[\mathbf{U} \otimes \mathbf{H} + \mathbf{H} \otimes \mathbf{U}] + p\mathbf{g}, \tag{52}$$

$$\mathbf{T}^{EM} \equiv \frac{1}{4\pi}\left[\mathbf{F} \cdot \mathbf{F} - \frac{1}{4}(\mathbf{F{:}F})\mathbf{g}\right], \tag{53}$$

and recognize from equation (2) that

$$\frac{1}{c} \mathbf{J} \cdot \mathbf{F} = -\nabla_x \cdot \mathbf{T}^{EM}. \tag{54}$$

The multifluid equations of § 4 therefore reproduce the familiar equations of general relativistic hydrodynamics.

Note that the energy-momentum-stress tensor in equation (52) is the one for an ideal gas *with* heat flow. It does *not* contain terms for viscous momentum and energy transport, however. These latter terms would arise if we performed a more sophisticated treatment of the collision term. In addition, unlike nonrelativistic treatments (which require the second moment for the energy equation), the treatment here derives the conservation of energy using only the first moment. This occurs because the equations are relativistic and use the four-vector $\mathbf{u}$ in the first moment instead of the spatial velocity $\mathbf{v}$. The conservation of energy equation can be extracted by taking the component of equation (46) along the average velocity[4]

$$\mathbf{U} \cdot \nabla_x \varepsilon + (\varepsilon + p) \nabla_x \cdot \mathbf{U} = -\mathbf{j} \cdot \mathbf{F} \cdot \mathbf{U} - \nabla_x \cdot \mathbf{H} - \mathbf{H} \cdot (\mathbf{U} \cdot \nabla_x \mathbf{U})/c^2, \tag{55}$$

which is also known as the "first law of thermodynamics": the change in internal energy ε plus mechanical work is given by ohmic heating minus losses due to heat conduction.

5.2. *Charge Dynamics in Geometric Form*

A set of equations similar to (45) and (46) can be generated by weighting equations (35) and (39) with the particle charge q_a rather than the rest mass. The results are the equations of charge dynamics: the conservation of charge

$$\nabla_x \cdot \mathbf{J} = \nabla_x \cdot (\rho_q \mathbf{U} + \mathbf{j}) = 0 \tag{56}$$

(equivalent to eq. [8]) and the relativistic generalized Ohm's law

$$\nabla_x \cdot \mathbf{C} = \ell \left[\frac{1}{c}(\mathbf{U} + h\mathbf{j}) \cdot \mathbf{F} - \eta(\rho_q \mathbf{U} + \mathbf{j})\right], \tag{57}$$

where the charge-current-pressure tensor is given by

$$\mathbf{C} \equiv \left(\rho_q + \frac{\varepsilon_q + p_q}{c^2}\right) \mathbf{U} \otimes \mathbf{U} + \mathbf{U} \otimes \mathbf{j}' + \mathbf{j}' \otimes \mathbf{U} + p_q \mathbf{g}. \tag{58}$$

[3] In component notation, the inner and scalar products of two tensors are $[\mathbf{F \cdot G}]^{\alpha\beta} = F^{\alpha\nu} G_\nu^\beta$ and $\mathbf{F : G} = F^{\mu\nu} G_{\mu\nu}$.

[4] The Euler equations also can be extracted by projecting eq. (46) with the projection tensor $\mathbf{P}$, but they are of no interest in this paper.

Note the appearance of a beamed current $\mathbf{j}'$ *in the charge-current-pressure tensor while the source terms involve the unbeamed current* $\mathbf{j}$ *only.*

The individual charge-dynamical scalars are charge density, charge-weighted internal energy and pressure (per unit mass),

$$\rho_q \equiv \sum_a n_a q_a, \tag{59}$$

$$\varepsilon_q \equiv \sum_a \frac{q_a}{m_a}\varepsilon_a = \sum_a (n'_a - n_a) q_a c^2 \tag{60}$$
$$= \sum_a q_a c^2 \int_{\hat{\Omega}_u} \gamma(\gamma - 1)\aleph_a \, d\hat{\Omega}_u,$$

$$p_q \equiv \sum_a \frac{q_a}{m_a} p_a = \frac{1}{3}\sum_a q_a \int_{\hat{\Omega}_u} \gamma^2(\mathbf{v}\cdot\mathbf{v})\aleph_a \, d\hat{\Omega}_u, \tag{61}$$

and an enhancement γ_q in the spatial electric current due to relativistic beaming effects,

$$\mathbf{j}' \equiv \sum_a q_a \int_{\hat{\Omega}_u} \gamma^2 \mathbf{v} \aleph_a \, d\hat{\Omega}_u \equiv \sum_a q_a n_a \mathbf{V}'_a$$
$$\equiv \gamma_q \sum_a q_a n_a \mathbf{V}_a = \gamma_q \mathbf{j}. \tag{62}$$

Because the partial internal energies ε_a and pressures p_a have been defined previously (eqs. [48] and [50]), and because q_a and m_a are known, the quantities ε_q and p_q (eqs. [60] and [61]) are not new variables but rather different weightings of known equations of state. Only the six quantities ρ_q, $\mathbf{j}'$, and γ_q are new ones that need their own charge-dynamical equations. Five of those equations are, respectively, equation (56), the three components of equation (57) orthogonal to $\mathbf{U}$, and the component of equation (57) parallel to $\mathbf{U}$. Because $\mathbf{j}'$ is a four-vector, the sixth equation is a constraint on its components,

$$\mathbf{U}\cdot\mathbf{j}' = 0, \tag{63}$$

similar to equation (12).

The sources and sinks of current on the right-hand side of equation (57) are the Lorentz effect, the Hall effect, and resistive losses, with coefficients

$$\ell \equiv \sum_a (q_a^2 n_a / m_a), \tag{64}$$

$$h \equiv \frac{1}{\ell|\mathbf{j}|}\sum_a \frac{q_a}{m_a}|\mathbf{j}_a|, \tag{65}$$

$$\eta \equiv \frac{\nu}{\ell}, \tag{66}$$

where $|\mathbf{j}| \equiv (-\,\mathbf{j}\cdot\mathbf{j})^{1/2}$ is the magnitude of the spatial current.

Note that the definition of γ_q (eq. [62]) makes use of the fact that $\mathbf{V}'_a$ and $\mathbf{V}_a$ are essentially parallel, resulting in the current $\mathbf{j}$ being enhanced by an average Lorentz factor γ_q (which is ≥ 1). We can derive an equation for γ_q in a manner similar to that used for equation (55), arriving at

$$\mathbf{j}'\cdot\nabla_x(\gamma_q) = \gamma_q^2\Big\{\ell\eta\rho_q - \frac{1}{c^2}\big[\mathbf{j}'\cdot(\mathbf{U}\cdot\nabla_x\mathbf{U})$$
$$+ \mathbf{U}\cdot\nabla_x\varepsilon_q + (\varepsilon_q + p_q)\nabla_x\cdot\mathbf{U}\big]\Big\}$$
$$- \gamma_q(\gamma_q - 1)\nabla_x\cdot\mathbf{j}' - \frac{\gamma_q h\ell}{c^3}\mathbf{j}'\cdot\mathbf{F}\cdot\mathbf{U}. \tag{67}$$

While analogous to equation (55), equation (67) is of a very different character. Everything in it, including $\mathbf{j}'$, ρ_q, $\mathbf{U}$, $\mathbf{F}$, and even ε_q and p_q can be considered known. What remains is a current-weighted gradient of the Lorentz factor equaling a quadratic function of that Lorentz factor. In the rest frame of the fluid (or when the fluid is at rest), the gradient $\mathbf{j}'\cdot\nabla_x$ loses all time dependence, and *the equation takes on an elliptical character*. The distribution of γ_q must be solved implicitly on each hypersurface, employing appropriate boundary conditions, etc. Equation (67) can be thought of as a constraint on the current beaming factor γ_q, enforcing the conservation of charge-weighted internal energy flow via $\mathbf{j}'$ at the same time as conservation of charge flow via $\mathbf{j}$ is enforced by equation (56).

Implicit equations for Lorentz factors are not unusual in relativistic hydrodynamics or magnetohydrodynamics (Duncan & Hughes 1994; Martí & Müller 1999;[5] Hughes, Miller, & Duncan 2002; Koide 2003). In most cases, however, they are simple algebraic equations that need to be solved in a single cell in spacetime at each time step. In this case, the equation contains gradient information on γ_q, and therefore must be solved over the entire hypersurface simultaneously. When the fluid is not at rest with respect to the frame in which the gradient ∇_x is computed, and is flowing relativistically, $\mathbf{j}'$ can have a significant temporal component, rendering equation (67) a hyperbolic equation. Nevertheless, computationally, it would be wise to solve this particular equation implicitly at all time steps in order to avoid numerical problems when the local fluid velocity slows down.

5.3. *Charge Dynamics in Component Form*

The equations of charge dynamics (56)–(58) are valid in any frame. Therefore, we can immediately convert them to component form in any metric. Using ρ_q, $\mathbf{j}'$, and γ_q as the variables, they are

$$\big[(\rho_q \mathrm{U}^\mu + \mathrm{j}'^\mu/\gamma_q)\sqrt{-g}\big]_{,\mu} = 0, \tag{68}$$

$$(\mathrm{C}^{\alpha\mu}\sqrt{-g})_{,\mu} + \Gamma^\alpha_{\mu\nu}\mathrm{C}^{\mu\nu}\sqrt{-g}$$
$$= \ell\left[\frac{1}{c}\left(\mathrm{U}^\mu + \frac{h\mathrm{j}'^\mu}{\gamma_q}\right)\mathrm{F}^\alpha_\mu - \eta\left(\rho_q\mathrm{U}^\alpha + \frac{\mathrm{j}'^\alpha}{\gamma_q}\right)\right]\sqrt{-g}, \tag{69}$$

$$\mathrm{j}'^\mu \mathrm{U}^\nu \mathrm{g}_{\mu\nu} = 0, \tag{70}$$

with the charge-current-pressure tensor given by

$$\mathrm{C}^{\alpha\beta} = \big[\rho_q + (\varepsilon_q + p_q)/c^2\big]\mathrm{U}^\alpha\mathrm{U}^\beta$$
$$+ \mathrm{U}^\alpha \mathrm{j}'^\beta + \mathrm{j}'^\alpha \mathrm{U}^\beta + p_q \mathrm{g}^{\alpha\beta}, \tag{71}$$

and the usual Einstein summation convention and comma derivative applying,

$$\mathrm{j}'^\mu \gamma_{q,\mu} \equiv \sum_{\mu=0}^{3} \mathrm{j}'^\mu \frac{\partial\gamma_q}{\partial x^\mu}.$$

In addition, we can replace the zeroth (temporal) component of equation (69) with that projected along the four-velocity

[5] Martí & Müller (1999) available at http://www.livingreviews.org/lrr-1999-3.

to get a component form of equation (67) for the current beaming factor,

$$\mathrm{j}'^{\mu}\gamma_{q,\mu} = \gamma_q^2\Bigg\{\ell\eta\rho_q - \frac{1}{c^2}\Bigg[\mathrm{j}'^{\mu}\mathrm{g}_{\mu\nu}\mathrm{U}^{\lambda}(\mathrm{U}^{\nu}_{,\lambda} + \Gamma^{\nu}_{\sigma\lambda}\mathrm{U}^{\sigma}) + \mathrm{U}^{\mu}\varepsilon_{q,\mu} + \frac{(\varepsilon_q + p_q)}{\sqrt{-g}}(\mathrm{U}^{\mu}\sqrt{-g})_{,\mu}\Bigg] - \frac{\gamma_q(\gamma_q - 1)}{\sqrt{-g}}(\mathrm{j}'^{\mu}\sqrt{-g})_{,\mu} - \frac{\gamma_q h\ell}{c^3}\mathrm{j}'^{\mu}\mathrm{F}_{\mu\nu}\mathrm{U}^{\nu}\Bigg\}. \quad (72)$$

6. DISCUSSION

6.1. *Special Cases*

It is useful to examine a few special cases of relativistic charge dynamics to compare with previous authors' work.

6.1.1. *Steady State with No Hall Term*

Under many conditions the terms on the left-hand side of equation (57) are small compared to those on the right hand side, because the time and length scales over which plasma properties vary are long. In addition, the Hall effect is often small compared to the Lorentz and resistive effects. The remaining terms, when projected orthogonal to **U**, then reduce to equation (14). They can be reduced further to the time-independent relativistic and nonrelativistic forms in § 2 under conditions of infinite conductivity, subrelativistic flow, etc.

6.1.2. *Cold Plasma and Nonrelativistic Flow*

Equation (57) also can be reduced to the form of equation (17) if we make the following cold plasma assumptions: (1) the charge-weighted internal energy and pressure are negligible compared with $\rho_q c^2$, and (2) the current beaming factor $\gamma_q = 1$ so that $\mathbf{j}' = \mathbf{j}$. These conditions allow the charge-current-pressure tensor to be rewritten as

$$\mathbf{C} \equiv \mathbf{U}\otimes\mathbf{J} + \mathbf{J}\otimes\mathbf{U} - \rho_q\mathbf{U}\otimes\mathbf{U} + p_q\mathbf{g}$$

and allow us to ignore the temporal component of equation (57), which is now redundant with the conservation of charge equation. With the assumption that the metric is that of Minkowskian flat space (with nonzero components $\mathrm{g}^{\alpha\alpha} = [-1,\ 1,\ 1,\ 1]$, no sum on α), the three spatial components of equation (57) (those projected orthogonal to $\mathbf{e}_t$, *not* $\mathbf{e}_\mathrm{U}$) reduce to equation (17).

If we additionally make the following nonrelativistic assumptions that $\mathbf{U} = [c,\ V_x,\ V_y,\ V_z]$ and $|\boldsymbol{V}| \ll c$, so that the orthogonal current $\mathbf{j}$ is approximately the spatial current $\boldsymbol{J} = [0,\ J_x,\ J_y,\ J_z]$, then equation (57) reduces to equation (16).

6.1.3. *Relativistic Pair Plasma*

For highly relativistic flows a time-dependent form of equation (57) will be needed. However, for certain plasmas, such as a relativistic pair plasma near a black hole, equation (67) can still be simplified somewhat. In this case, $m_- = m_+$ and $q_- = -q_+$, so $\varepsilon_q = p_q = 0$, and the Hall coefficient vanishes explicitly. Then equation (67) reduces to

$$\mathbf{j}'\cdot\nabla_x(\gamma_q) = \gamma_q^2[\ell\eta\rho_q - \mathbf{j}'\cdot(\mathbf{U}\cdot\nabla_x\mathbf{U})/c^2] - \gamma_q(\gamma_q - 1)\nabla_x\cdot\mathbf{j}'. \quad (73)$$

In other words, the beaming factor distribution is determined by the competition between local "creation" of charge by collisions and the loss of charge through the beamed current $\mathbf{j}'$.

6.1.4. *Uniform Adiabatic Index and Mixture*

More generally, if the plasma is made up of particle partial fluids that have the same adiabatic index, and if that index and the fractional pressure $\pi_a \equiv p_a/p$ of each species are uniform throughout Ω_x, then we have each quantity ε_a, ε, and ε_q related to their respective pressures by the simple relation

$$\varepsilon_i = \frac{1}{\Gamma - 1}p_i, \quad (74)$$

where $i = a$, q, or blank, and Γ is the adiabatic index. Then the ratio of charge-weighted to total pressure (and that for internal energy) is a uniform constant throughout Ω_x,

$$\frac{p_q}{p} = \frac{\varepsilon_q}{\varepsilon} = \sum_a \frac{q_a}{m_a}\pi_a \equiv \zeta. \quad (75)$$

We then can use equation (55) to eliminate the charge-weighted thermodynamic quantities in equation (67) to obtain a simpler equation for γ_q,

$$\mathbf{j}'\cdot\nabla_x(\gamma_q) = \gamma_q^2\left[\ell\eta\rho_q - \left(\mathbf{j}' - \frac{\zeta\mathbf{H}}{c^2}\right)\cdot\frac{\mathbf{U}\cdot\nabla_x\mathbf{U}}{c^2} - \nabla_x\cdot\left(\mathbf{j}' - \frac{\zeta\mathbf{H}}{c^2}\right)\right] + \gamma_q\left[\nabla_x\cdot\mathbf{j}' - \frac{(h\ell - \zeta)}{c^3}\mathbf{j}'\cdot\mathbf{F}\cdot\mathbf{U}\right]. \quad (76)$$

The pair plasma is a special case of these conditions, and equation (76) reduces to equation (73) in this case (ζ, $h \to 0$). Equation (76) is useful for showing how the equation for γ_q is decoupled from the terms involving ε_q in equation (67). The quantity $\mathbf{j}' - \zeta\mathbf{H}/c^2$ is a residual (beamed) current whose properties can affect the value of the current beaming factor γ_q.

6.2. *One-Fluid versus Multifluid Theory*

The advantage of one-fluid theory, of course, is that, by summing the multifluid equations and deriving thermodynamic quantities that close the system, the many equations for each particle species are reduced to only five (eqs. [45] and [46]). However, with the introduction of a set of five new one-fluid equations (eqs. [56] and [57]), one must ask whether it is still useful to deal with the one-fluid equations rather than the more instructive multifluid equations, particularly if there are only two fluids (ions and electrons).

The answer still appears to be a qualified "yes," although it is quite likely that the multifluid equations will become more important in the next few decades, if not sooner. First of all, most current astrophysical codes (mainly hydrodynamic and magnetohydrodynamic) are one-fluid codes. While nevertheless a significant task, the addition of the one-fluid charge dynamical equations to existing MHD codes is still much less effort than developing an entirely new multifluid code. Second, even when constructing a new code that involves only ions and electrons, developing that two-fluid code will be a much greater task than developing a one-fluid code with the 10 equations of hydro- and charge dynamics. In one-fluid theory, the collision terms can be treated with much less rigor than in a two-fluid code. In the former, because thermodynamic equilibrium is assumed, one need only postulate an approximate resistivity, as we have done in § 5. In the latter case, however, one must carefully handle collisional momentum and energy transfer between each species, as well as the scattering of particles by electromagnetic oscillations with wavelengths shorter than a cell size. Otherwise, the simulated multitemperature structure of the fluid will be meaningless. Finally, as is the case in simulations of late stages of stellar evolution and collapse, when dealing with the collapse of dense matter to black holes, there probably will be many more than two species of particle (neutrons, protons, electrons, positrons, heavy iron-peak nuclei, etc.). With five multifluid equations for each species, the number of equations to integrate could be significantly greater than the 10 needed for hydro- and charge dynamics.

Therefore, any multifluid astrophysical codes that are to be developed in the next few years are likely to be two-fluid only and probably will initially make the assumption of thermodynamic equilibrium. In that case, their results will be similar to those obtained by older MHD codes that have been modified to handle charge dynamics. Nevertheless, these new codes will become increasingly sophisticated as more particle physics is added and should lead to a greater understanding of black hole formation than is possible from one-fluid simulations alone.

7. CONCLUSIONS

This paper has used geometric frame-independent tensor notation to derive what the author believes is the first set of one-fluid equations to be published that correctly describes the flow of charge in general relativistic environments. Previously used or suggested approximations (e.g., force-free field, magnetohydrodynamics, even currently available relativistic forms of the generalized Ohm's law) do not correctly describe black hole astrophysics on timescales much shorter than the collapse time ($\ll GM/c^3$) or in strong gravitational fields. The principle equations of charge dynamics are equations (56) and (57), with an alternative form for the relativistic current equation (67). Proper handling of charge flow in such environments will be important for understanding the details of highly relativistic astrophysical events like black hole formation and relativistic jet generation, which may be important for understanding gamma-ray bursts, etc. (Multifluid equations are also derived, but the collision terms are not treated with sufficient rigor in this paper to make them useful for detailed simulations at this time.) These equations of "charge dynamics," also given in component form in equations (68), (69), and (72), are valid for any flow velocity (causal), in any reference frame (covariant), and in any spacetime metric (general relativistic). They are therefore suitable not only for flow in stationary metrics such as Schwarzschild or Kerr, but also for general numerical relativity calculations that include a fluid with an embedded electromagnetic field. The equations of charge dynamics were shown to reduce to a variety of prior "generalized Ohm's laws" in the appropriate cold-plasma, flat-space, or nonrelativistic limits.

In addition to the general relativistic nature of the equations, the principal difference between this paper and previous work is that it does *not* make the assumption of a cold plasma. Not only must one deal with quantities such as charge-weighted internal energy and pressure, one also must solve for the current beaming factor γ_q that distinguishes the beamed current $\mathbf{j}'$ from the unbeamed $\mathbf{j}$. The equation for γ_q, generated by the subtraction of the energy equations for positive and negative ions, is primarily implicit and global in character, providing a constraint on the flow of charge. This result is in sharp contrast to the assertion (Khanna 1998) that a one-fluid theory is only possible for a cold plasma. The charge-current tensor approach produces a fourth "energy" equation for the beamed current that can take into account fast current drift velocities and hot plasma as well as fast bulk speeds.

While of course valid in more benign environments, the charge-dynamical form of Ohm's law probably will be needed only in very violent environments such as electromagnetic, rotating black hole *formation*. These events can have significant fluid and metric shear that may affect charge flow on timescales shorter than the current rise time. However, it is precisely these environments that are believed to obtain in "collapsars," which form in the centers of massive stars, and in mergers of neutron stars in close binary systems. And it is, these systems that are believed to lead directly to the formation of electromagnetic jets and their associated observable events. Charge dynamics, therefore, may play a significant role in understanding the engine that generates the highly relativistic jets seen in gamma-ray bursts and other sources.

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APPENDIX

VELOCITY INTEGRALS IN THREE-SPACE

The velocity integrals in §§ 4 and 5 were cast as being over the four-dimensional volume $\hat{\Omega}_u$. This causes each integral, even those for quantities that appear in nonrelativistic dynamics (n_a, p_a, ε_a), to contain a Lorentz factor (eq. [30]) in its kernel. In this Appendix we show that each can be converted to its more familiar, three-space form by confining the integration to take place on the mass hyperboloid only.

We begin by defining a three-dimensional density

$$f_a \equiv \frac{\partial^6 \mathrm{N}_a}{m_a^3 \partial \Upsilon_u \partial \Upsilon_x} = \frac{\partial^6 \mathrm{N}_a}{\partial \Upsilon_p \partial \Upsilon_x} \tag{A1}$$

as the distribution of particles in three-momentum $\boldsymbol{p}$, where m_a is the particle rest mass and $d\Upsilon_p = m_a^3 d\Upsilon_u$. This distribution is related to $\aleph_a$ by a delta function that enforces the mass hyperboloid,

$$\aleph_a = m_a^3 f_a \delta\left(\epsilon - \frac{\gamma^2 c}{2}\right) = m_a^3 f_a \frac{\delta(\mathrm{u}^{\hat{0}} - \gamma c)}{\mathrm{u}^{\hat{0}}/c} \tag{A2}$$

(see eq. [20]).

The four-volume integral of any kernel K, weighted by γ, now can be replaced by a three-integral over f_a on the mass shell, with no γ weighting:

$$\begin{aligned}\int_{\hat{\Omega}_u} \gamma \aleph_a K(\mathbf{x}, \mathbf{u}) d\hat{\Omega}_u &= \int_{\Upsilon_u} \int_{\mathrm{u}^{\hat{0}}} \gamma m_a^3 f_a \frac{\delta(\mathrm{u}^{\hat{0}} - \gamma c)}{\mathrm{u}^{\hat{0}}/c} K(\mathbf{x}, \mathrm{u}^{\hat{0}}, \boldsymbol{p})\, d\hat{\Omega}_u \\ &= \int_{\Upsilon_p} f_a d^3 p \int_{\mathrm{u}^{\hat{0}}} \frac{\gamma}{\mathrm{u}^{\hat{0}}/c} \delta(\mathrm{u}^{\hat{0}} - \gamma c) K(\mathbf{x}, \mathrm{u}^{\hat{0}}, \boldsymbol{p})\, d\mathrm{u}^{\hat{0}} \\ &= \int_{\Upsilon_p} K(\mathbf{x}, \gamma c, \boldsymbol{p}) f_a\, d^3 p. \end{aligned} \tag{A3}$$

Equations (36), (37), (48), and (50) now take on their familiar forms,

$$n_a = \int_{\Upsilon_p} f_a\, d^3 p, \tag{A4}$$

$$\mathbf{V}_a = \frac{1}{n_a} \int_{\Upsilon_p} \mathbf{v} f_a\, d^3 p, \tag{A5}$$

$$\varepsilon_a = \int_{\Upsilon_p} (\gamma - 1) m_a c^2 f_a\, d^3 p, \tag{A6}$$

$$p_a = \frac{1}{3} \int_{\Upsilon_p} \boldsymbol{p} \cdot \mathbf{v} f_a\, d^3 p. \tag{A7}$$

[Recall that in the local Lorentz frame of the fluid $\mathbf{v} = (0, \mathrm{v}^1, \mathrm{v}^2, \mathrm{v}^3)$.] However, $\mathbf{V}'_a$ in equation (41), which is used to construct the beamed current $\mathbf{j}'$, has no nonrelativistic analog (other than $\mathbf{j}$ itself when $|\mathbf{v}| \ll c$), and therefore must always involve the Lorentz factor

$$\mathbf{V}'_a = \frac{1}{n_a} \int_{\Upsilon_p} \gamma \mathbf{v} f_a\, d^3 p. \tag{A8}$$

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